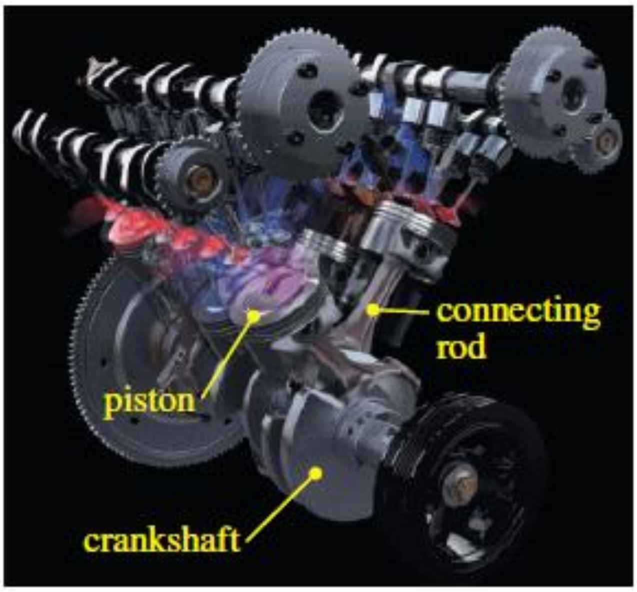
Rigid Body Dynamics: Planar Motion

Thursday, April 3, 2025 9:46 AM



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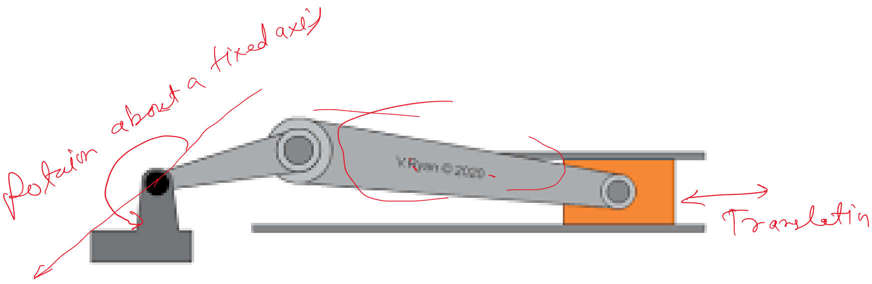
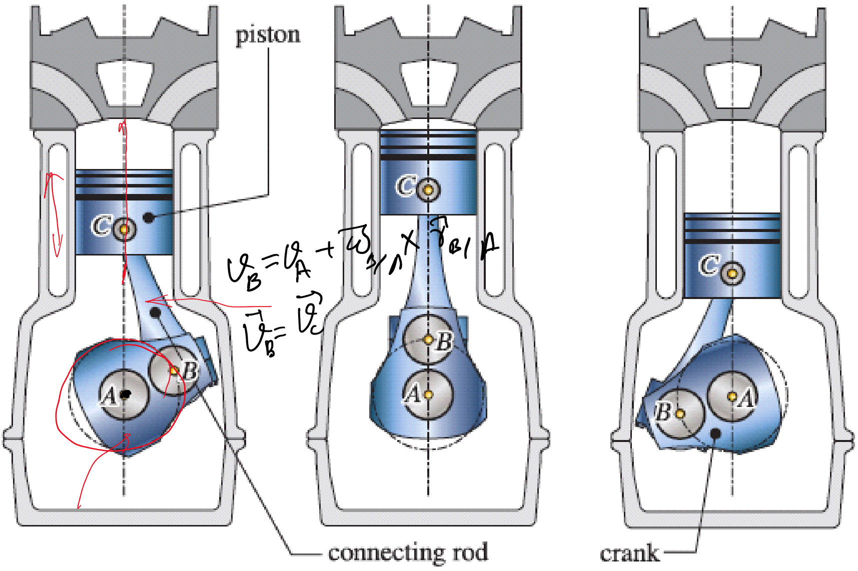
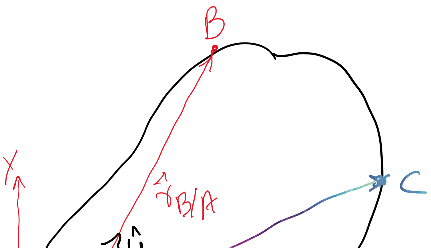




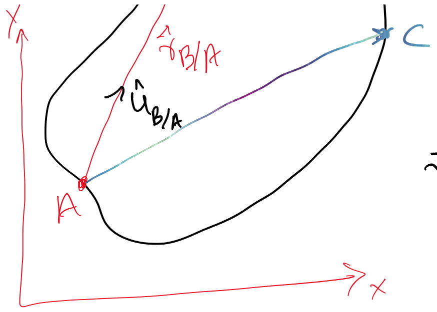
$v_B = v_A + \vec{\omega}_{B/A} \times \vec{r}_{B/A}$

$\vec{v}_B = \vec{v}_C$



$\vec{r}_{B/A} = r_{B/A}\,\hat{u}_{B/A}$

$\dot{r}_{B/A} = 0$



$$\dot{r}_{B/A} = 0$$

$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$$

$$\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A$$

$$= \frac{d\vec{r}_{B/A}}{dt}$$

$$= r_{B/A}\,\dot{\hat{u}}_{B/A}$$

$$\vec{v}_B - \vec{v}_A = r_{B/A}\left(\vec{\omega}_{B/A} \times \hat{u}_{B/A}\right)$$

$$\boxed{\vec{v}_B = \vec{v}_A + \vec{\omega}_{B/A} \times \vec{r}_{B/A}}$$

$$\boxed{\vec{v}_C = \vec{v}_A + \vec{\omega}_{C/A} \times \vec{r}_{C/A}}$$

$$\vec{\omega}_{B/A} = \vec{\omega}_{C/A} = \vec{\omega}_{Body}$$

$$\boxed{\vec{v}_B = \vec{v}_A + \vec{\omega}_{Body} \times \vec{r}_{B/A}}$$

$$\vec{a}_B = \vec{a}_A + \dot{\vec{\omega}}_{Body} \times \vec{r}_{B/A} + \vec{\omega}_{Body} \times \dot{\vec{r}}_{B/A}$$

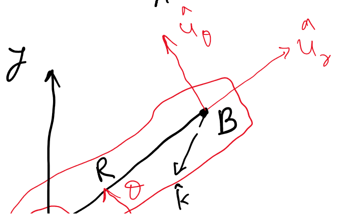
$$= \vec{a}_A + \vec{\alpha}_{Body} \times \vec{r}_{B/A} + \vec{\omega}_{Body} \times \left(\vec{\omega}_{Body} \times \vec{r}_{B/A}\right)$$

Translation

$$\left.\begin{aligned} \vec{\omega}_{Body} &= 0 \\ \alpha_{Body} &= 0 \end{aligned}\right\} \quad \begin{aligned} \vec{v}_B &= \vec{v}_A \\ \vec{a}_B &= \vec{a}_A \end{aligned}$$

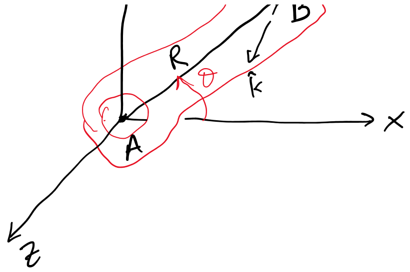
Rotation about a Fixed axis

$$\left.\begin{aligned} \vec{v}_A &= 0 \\ \vec{a}_A &= 0 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \vec{v}_B &= \vec{\omega}_{Body} \times \vec{r}_{B/A} \\ \vec{a}_B &= \vec{\alpha}_{Body} \times \vec{r}_{B/A} + \vec{\omega}_{Body} \times \left(\vec{\omega}_{Body} \times \vec{r}_{B/A}\right) \end{aligned}$$



$$\vec{\omega}_{Body} = \dot{\theta}\,\hat{k}$$

$$\vec{\alpha}_{Body} = \ddot{\theta}\,\hat{k}$$



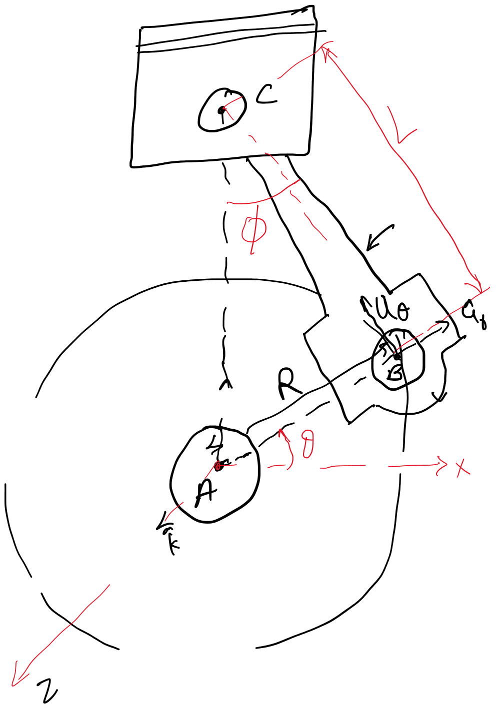
$\omega_{Body} =$

$$\vec{v}_B = \dot{\theta}\hat{k} \times R\hat{u}_r = R\dot{\theta}\hat{u}_\theta$$

$$\vec{a}_B = R\ddot{\theta}\hat{u}_\theta - R\dot{\theta}^2\hat{u}_r$$

Velocity Analysis

Vector Approach



$$\vec{v}_C = \vec{v}_B + \vec{\omega}_{BC} \times \vec{r}_{C/B}$$

$$\vec{\omega}_{BC} = \omega_{BC}\hat{k}$$

$$\vec{r}_{C/B} = L(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})$$

$$\vec{v}_C = v_{cy}\hat{j}$$

$$\vec{v}_B = \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A}$$

$$= \omega_{AB}\hat{k} \times R\hat{u}_r$$

$$= \omega_{AB}R(\hat{k} \times \hat{u}_r)$$

$$= \omega_{AB}R\hat{u}_\theta$$

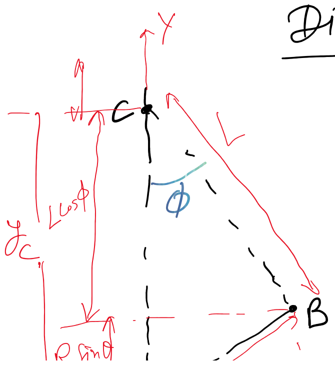
$$= \omega_{AB}R(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$

$$v_{cy}\hat{j} = -\left(R\omega_{AB}\sin\theta + \omega_{BC}\sqrt{L^2 - R^2\cos^2\theta}\right)\hat{i} + R(\omega_{AB} - \omega_{BC})\cos\theta\,\hat{j}$$

$v_{cy} = ?$

$\omega_{BC} = ?$

Differentiation of Constraint

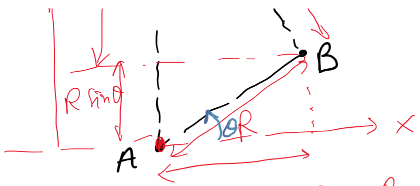


$y_c = ?$

$v_c = \dot{y}_c$

$$y_c = R\sin\theta + L\cos\phi$$

$$\dot{y}_c = R(\cos\theta)\dot{\theta} - L(\sin\phi)\dot{\phi}$$



$$\dot{y}_c = R(\cos\theta)\dot{\theta} - L(\sin\phi)\dot{\phi}$$

$$v_{c_y} = R\omega_{AB}\left[1 + \frac{\sin\theta}{\sqrt{\left(\frac{L}{R}\right)^2 - \cos^2\theta}}\right]\cos\theta$$

$$L\sin\phi = R\cos\theta$$

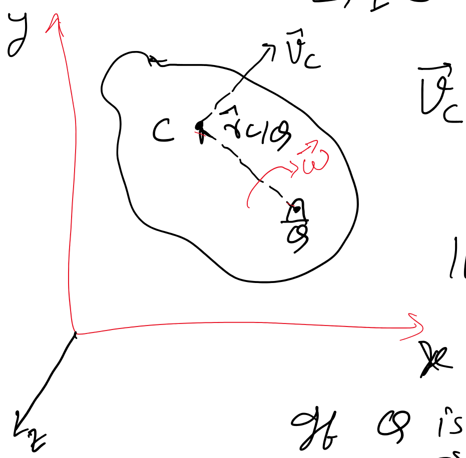
$$\sin\phi = \frac{R}{L}\cos\theta$$

$$\cos\phi = \frac{\sqrt{L^2 - R^2\cos^2\theta}}{L}$$

Instantaneous Centre

↳ IC of Rotation

↳ IC of zero velocity



$$\vec{v}_C = \cancel{\vec{v}_Q}^{\,0} + \vec{\omega}_{body} \times \vec{r}_{C/Q}$$

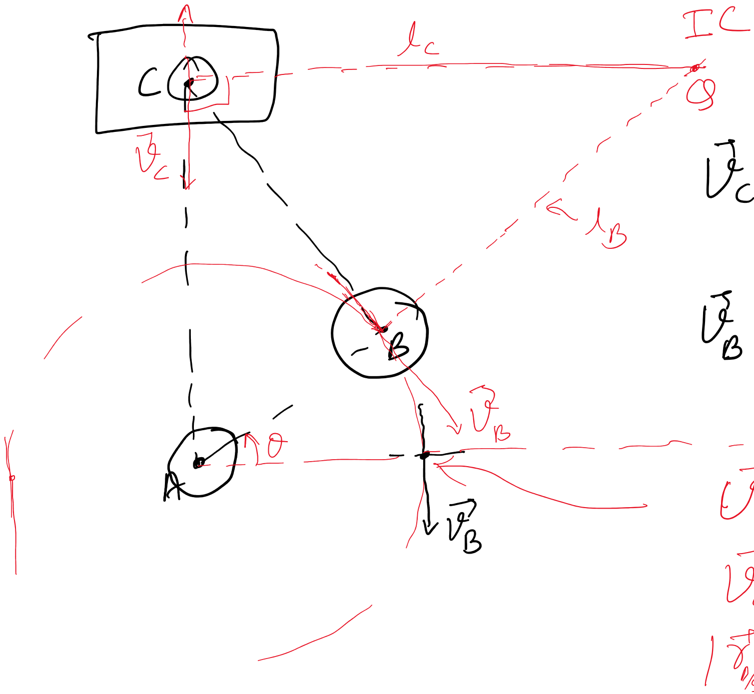
$$= \vec{\omega}_{body} \times \vec{r}_{C/Q}$$

$$|\vec{v}_C| = \omega_{body}|\vec{r}_{C/Q}|$$

If Q is fixed

$\Rightarrow \vec{v}_Q = 0$

$Q \equiv IC$



$$\vec{v}_C = \vec{v}_Q + \vec{\omega}_{body} \times \vec{r}_{C/Q}$$

$$= \vec{\omega}_{BC} \times \vec{r}_{C/Q}$$

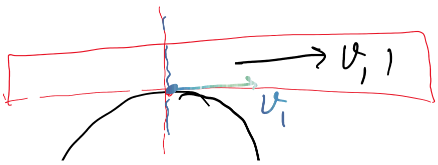
$$\vec{v}_B = \vec{\omega}_{BC} \times \vec{r}_{B/Q}$$

$$\vec{v}_B = \vec{\omega}_{BC} \times \vec{r}_{B/Q}$$

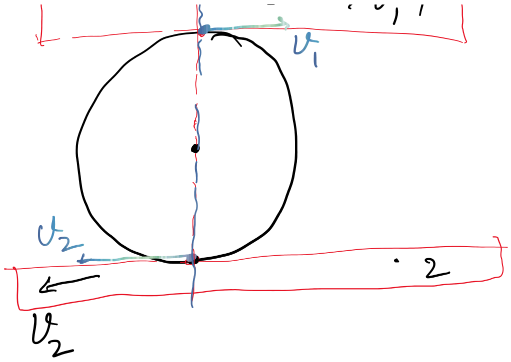
$$\vec{v}_C = \vec{\omega}_{BC} \times \vec{r}_{C/Q}$$

$$|\vec{r}_{B/Q}| \; \& \; |\vec{r}_{C/Q}| \longrightarrow \infty$$

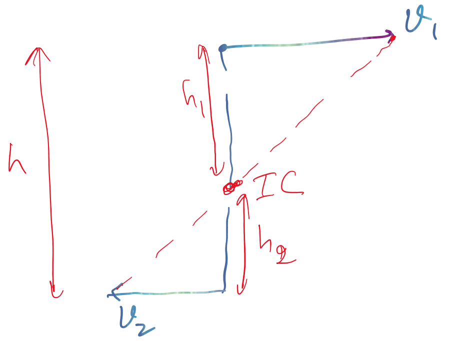
$$\omega_{BC} = 0$$



wheel rotates without slip



wheel rotates without slip



$$\frac{v_1}{h_1} = \frac{v_2}{h_2} = \omega_W$$

Acceleration Analysis

Vector Approach

$$\vec{a}_C = \vec{a}_B + \vec{\alpha}_{BC} \times \vec{r}_{C/B} + \vec{\omega}_{BC} \times (\vec{\omega}_{BC} \times \vec{r}_{CB})$$

$$\vec{a}_C = \vec{a}_B + \vec{\alpha}_{BC} \times \vec{r}_{C/B} - \omega_{BC}^2 \vec{r}_{CB}$$

Differentiation Approach

$$y_C = R\sin\theta + L\cos\phi$$

$$\ddot{y}_C = a_{Cy} = R\ddot{\theta}\cos\theta - R\dot{\theta}^2\sin\theta - L\ddot{\phi}\sin\phi - L\dot{\phi}^2\cos\phi$$

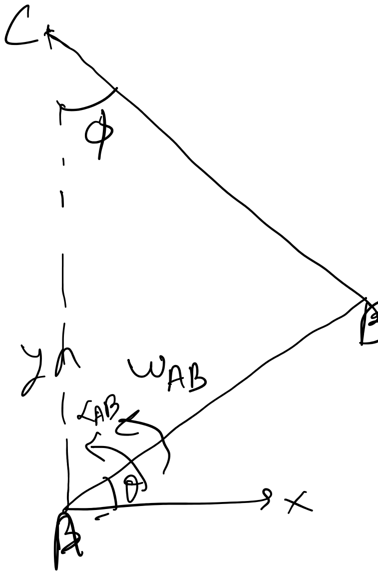
$$\vec{a}_C = \vec{a}_B + \vec{a}_t + \vec{a}_n$$

$$-\omega_{body}^2 \vec{r}_{C/B}$$

No IC of acc^n

C

$$\vec{a}_C = \vec{a}_B + \vec{\alpha}_{BC} \times \vec{r}_{C/B} - \omega_{BC}^2 \vec{r}_{C/B}$$



$$\vec a_C = \vec a_B + \vec\alpha_{BC}\times \vec r_{C/B} - \omega_{BC}^2\, \vec r_{C/B}$$

$$\vec\alpha_{BC} = \alpha_{BC}\hat k$$

$$\omega_{BC} = \frac{-\omega_{AB}\sin\theta}{\sqrt{\left(\frac{L}{R}\right)^2-\cos^2\theta}}$$

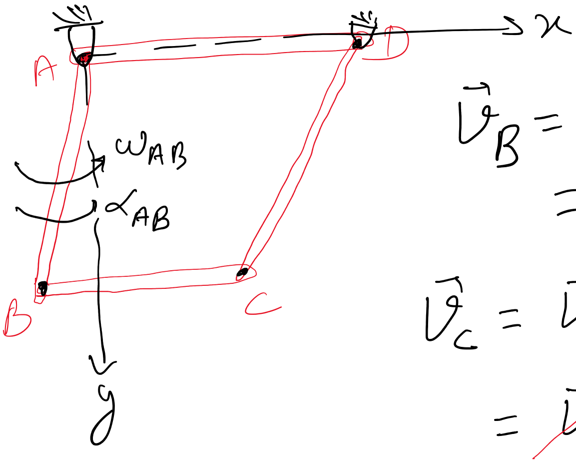
$$\vec r_{C/B} = -L\sin\phi\,\hat i + L\cos\phi\,\hat j$$

$$\sin\phi = \frac{R}{L}\cos\theta$$

$$\vec a_B = \cancel{\vec a_A}^{\,0} + \vec\alpha_{AB}\times\vec r_{B/A} - \omega_{AB}^2\,\vec r_{B/A}$$

$$a_{C_x} = 0$$

$$a_{C_y} = \ ?$$



$$\vec v_B = \vec v_A + \vec\omega_{AB}\times\vec r_{B/A}$$
$$= \omega_{AB}\left(\hat k\times\vec r_{B/A}\right)$$

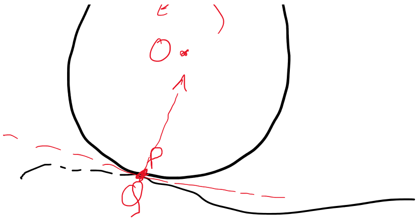
$$\vec v_C = \vec v_B + \omega_{BC}\left(\hat k\times\vec r_{C/B}\right)$$
$$= \cancel{\vec v_D}^{\,0} + \omega_{CD}\left(\hat k\times\vec r_{C/D}\right)$$
$$= \omega_{CD}\left(\hat k\times\vec r_{C/D}\right)$$

ω_{CD}, ω_{BC} in terms of ω_{AB}

α_{CD}, α_{BC} in terms of α_{AB}, ω_{AB}

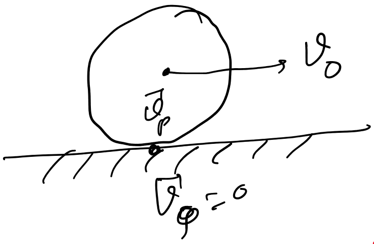
$$\vec v_{P/Q} = 0$$

$$\vec v_P = \vec v_Q$$



$$\vec{v}_P = \vec{v}_Q$$

$$\vec{v}_P = \vec{v}_O + \vec{\omega}_W \times \vec{r}_{P/O}$$



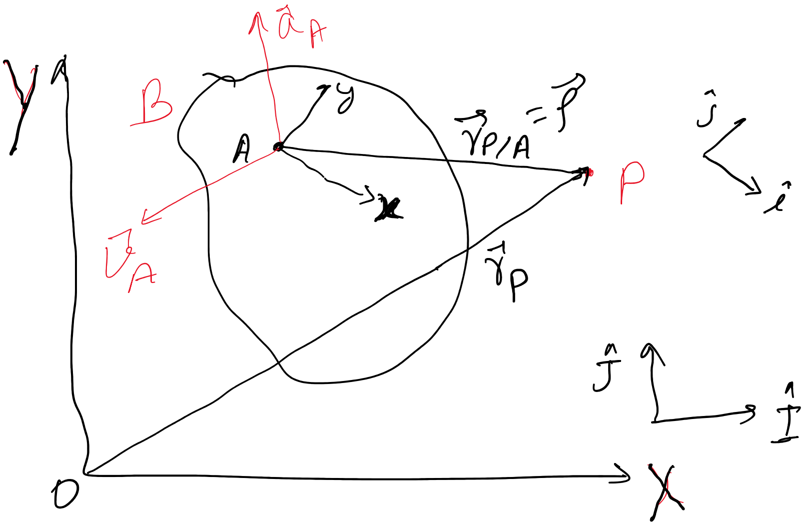
$$\vec{v}_P = \vec{v}_Q = 0$$

$$0 = v_O \hat{i} + \omega_W \hat{k} \times (-R\hat{j})$$

$$\boxed{\omega_W = -\frac{v_O}{R}}$$

$$\boxed{\alpha_W = -\frac{a_O}{R}}$$

Rotating Reference Frame



$$\vec{r}_P = \vec{r}_A + \vec{r}_{P/A} = \vec{r}_A + \vec{\rho}$$

$$\vec{r}_P = \vec{r}_A + (\rho_x \hat{i} + \rho_y \hat{j} + \rho_z \hat{k})$$

$$\vec{v}_P = \vec{v}_A + \dot{\rho}_x \hat{i} + \rho_x \dot{\hat{i}} + \dot{\rho}_y \hat{j} + \rho_y \dot{\hat{j}} + \dot{\rho}_z \hat{k} + \rho_z \dot{\hat{k}}$$

$$= \vec{v}_A + \dot{\vec{\rho}} + \rho_x (\vec{\Omega} \times \hat{i}) + \rho_y (\vec{\Omega} \times \hat{j}) \ldots$$

$$\vec v_P = \vec v_A + \dot{\vec\rho}$$

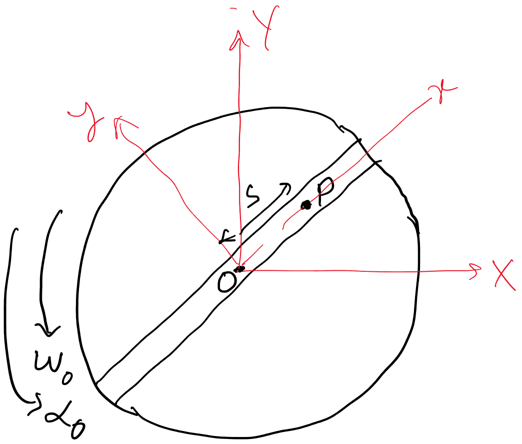
$$= \vec v_A + \dot{\vec\rho} + \rho_x(\vec\Omega\times\hat i) + \rho_y(\vec\Omega\times\hat j) + \rho_z(\vec\Omega\times\hat k)$$

$$\boxed{\vec v_P = \vec v_A + \dot{\vec\rho} + \vec\Omega\times\vec\rho}$$

$$\boxed{\vec v_P = \vec v_A + \underbrace{\vec v_{P_{rel}}}_{\text{Velocity of P w.r.t. moving frame}} + \vec\Omega\times\vec r_{P/A}}$$

$$\vec a_P = \vec a_A + \ddot\rho_x\hat i + \dot\rho_x\dot{\hat i} + \ddot\rho_y\hat j + \dot\rho_y\dot{\hat j} + \ddot\rho_z\hat k + \dot\rho_z\dot{\hat k}$$
$$+ \dot{\vec\Omega}\times\vec\rho + \vec\Omega\times\dot{\vec\rho}$$

$$\vec a_P = \vec a_A + \vec a_{P_{rel}} + 2\vec\Omega\times\vec v_{P_{rel}} + \dot{\vec\Omega}\times\vec r_{P/A} + \vec\Omega\times(\vec\Omega\times\vec r_{P/A})$$



$$\vec v_P = \vec v_O + \vec v_{P_{rel}} + \vec\Omega\times\vec r_{P/O}$$
$$= \vec 0 + \dot s\hat i + \omega_0\hat k\times s\hat i$$
$$\boxed{\vec v_P = \dot s\hat i + s\omega_0\hat j}$$

$$\vec a_P = \vec a_O + \vec a_{P_{rel}} + 2\vec\Omega\times\vec v_{P_{rel}} + \dot{\vec\Omega}\times\vec r_{P/O} + \vec\Omega\times(\vec\Omega\times\vec r_{P/O})$$
$$= (\ddot s - s\omega_0^2)\hat i + (s\alpha_0 + 2\dot s\omega_0)\hat j$$

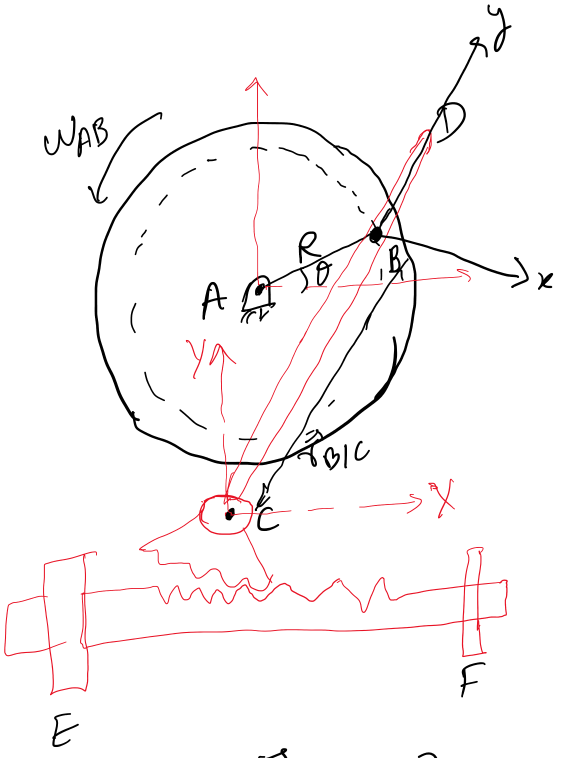
$$\vec v_P = \vec v_A + \vec v_{P_{rel}} + \vec\Omega\times\vec r_{P/A}$$

$$\vec a_P = \vec a_A + \vec a_{P_{rel}} + 2\vec\Omega\times\vec v_{P_{rel}} + \vec\alpha\times\vec r_{P/A} + \vec\Omega\times(\vec\Omega\times\vec r_{P/A})$$

$$\vec{a}_P = \vec{a}_A + \vec{a}_{P_{rel}} + 2\vec{\Omega}\times\vec{v}_{P_{rel}} + \vec{\alpha}\times\vec{r}_{P/A} + \vec{\Omega}\times(\vec{\Omega}\times r_{P/A})$$

Coriolis component of acceleration

$\boxed{\text{Gaspard-Gustave Coriolis}}$



$\vec{v}_{EF}$ & $\vec{a}_{EF}$?

$\vec{\omega}_{CD}$ & $\vec{\alpha}_{CD}$ = ?

$$\vec{v}_B = \vec{v}_C + \vec{v}_{B_{rel}} + \vec{\Omega}_{CD}\times\vec{r}_{B/C}$$

$$\vec{v}_B = \vec{\omega}_{AB}\times\vec{r}_{B/A}$$

$$= \omega_{AB}\hat{k}\times\left(r_{B/A}(\theta)\hat{i} + r_{B/A}(0)\hat{j}\right)$$

$$= -R\omega_{AB}\hat{i}$$

$$\vec{v}_C = 0$$

$$\vec{v}_{B_{rel}} = \dot{r}_{B/C}\hat{j}$$

$$\vec{\Omega}_{CD} = \Omega_{CD}\hat{k}$$

$$\vec{r}_{B/C} = r_{B/C}\hat{j}$$

$$-R\omega_{AB}\hat{i} = \dot{r}_{B/C}\hat{j} + \Omega_{CD}\hat{k}\times r_{B/C}\hat{j}$$

$\Omega_{CD} = ?$

$\omega_{CD} = ?$